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$\begin{cases} \text{discrete R.V : pmf} \\ \text{continuous R.V : pdf.} \end{cases}$

MATH 451/551

Chapter 3. Random Variables

3.3 Cumulative Distribution Variables

(CDF)

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Cumulative Distribution Functions

Cumulative Distribution Functions (CDF)

- **Distribution of a discrete random variable** X is characterized by its probability mass function $f(x)$ and its associated support $\mathcal{A}$.
- **Distribution of a continuous random variable** X is characterized by its probability density function $f(x)$ and its associated support $\mathcal{A}$.
- **The cumulative distribution function (cdf)** applies to both types of random variables $F(x) = P(X \leq x)$.

  R.V. value

  $F_X(x) = F(x) = P(X \leq x)$

  R.V. value value

  - **Discrete** random variable X

  $$F(x) = P(X \leq x) = \sum_{w \leq x} f(w).$$

  - **Continuous** random variable X

  $$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(w)dw.$$

Properties of CDF

impossible certainty.

- Since $F(x)$ is defined as a proobability, $0 \leq F(x) \leq 1$.

$F(x) = P(X \leq x)$

- $F(x)$ is a nondecreasing function of x, that is, for $a < b$, $F(a) \leq F(b)$.

① $F(a) = P(X \leq a)$

② $F(b) = P(X \leq b)$

$F(a) \leq F(b)$

- $\lim_{x \to -\infty} F(x) = 0$.
- $\lim_{x \to \infty} F(x) = 1$.
- $P(a < X \leq b) = F(b) - F(a) = P(X \leq b) - P(X \leq a)$.
- The random variables X and Y are identically distributed if and only if they have identical cumulative distribution functions.

$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$

① Discrete R.V. it is possible that $P(X = a) > 0$

$P(a < X \leq b) \neq P(a \leq X \leq b)$

② Continuous R.V. $P(X = a) = 0$ $P(a \leq X \leq b) = P(a < X \leq b)$

Example 1

Flip a fair coin twice. Let X be the number of heads tossed. Find $F(x)$.

$\mathcal{A} = \{0, 1, 2\} \Rightarrow X$ is Discrete.

$S = \{\underbrace{(H,H)}_{\downarrow}, \underbrace{(H,T), (T,H)}_{\downarrow}, \underbrace{(T,T)}_{\downarrow}\}$

$X = \{\ 2\ ,\ 1\ ,\ 0\ \}$

$$f_{\underset{\uparrow \text{R.V}}{X}}(x) = f(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{2}{4} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases}$$

$$F_X(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{4} + \frac{2}{4} & x = 1 \\ \frac{1}{4} + \frac{2}{4} + \frac{1}{4} & x = 2 \end{cases} = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{3}{4} & x = 1 \\ 1 & x = 2 \end{cases}$$

Example 2

Find the cumulative distribution function for a continuous random variable X that is uniformly distributed between 0 and 1.

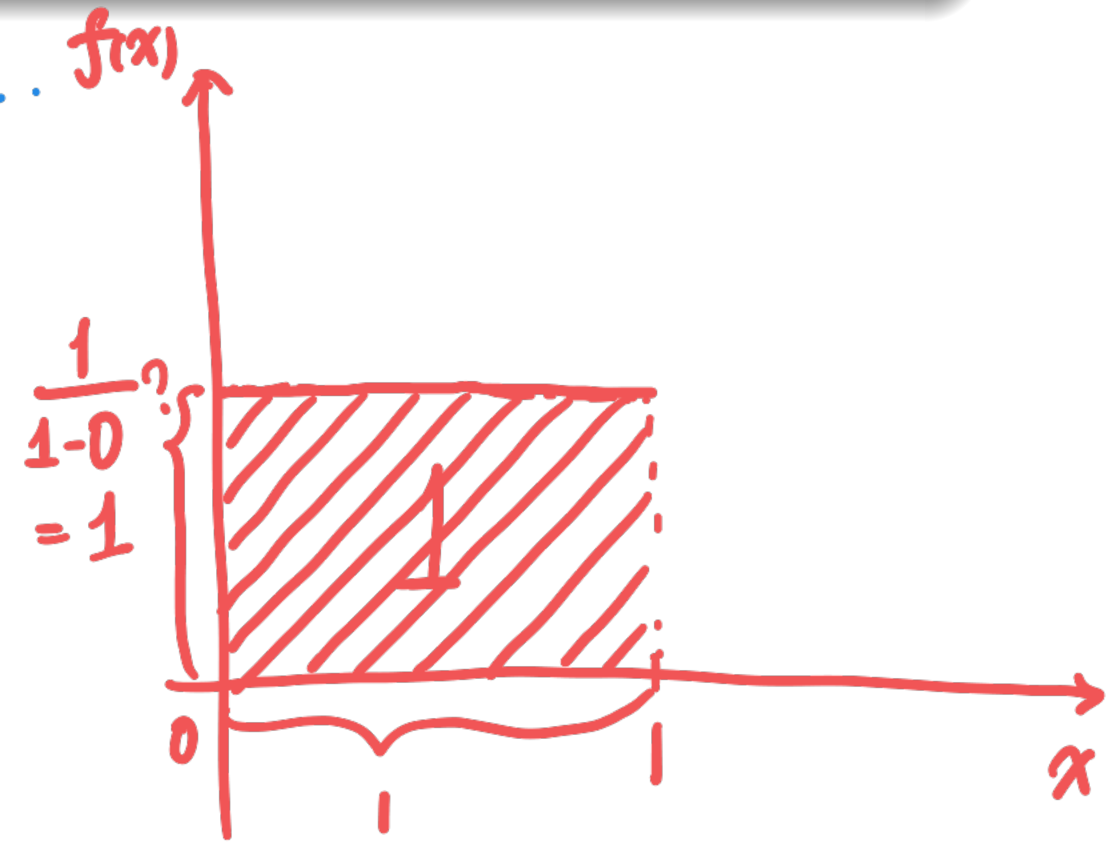
$\mathcal{S} = \{0 < x < 1\} \Rightarrow X$ is continuous.

$f_X(x) = 1 \qquad 0 < x < 1$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

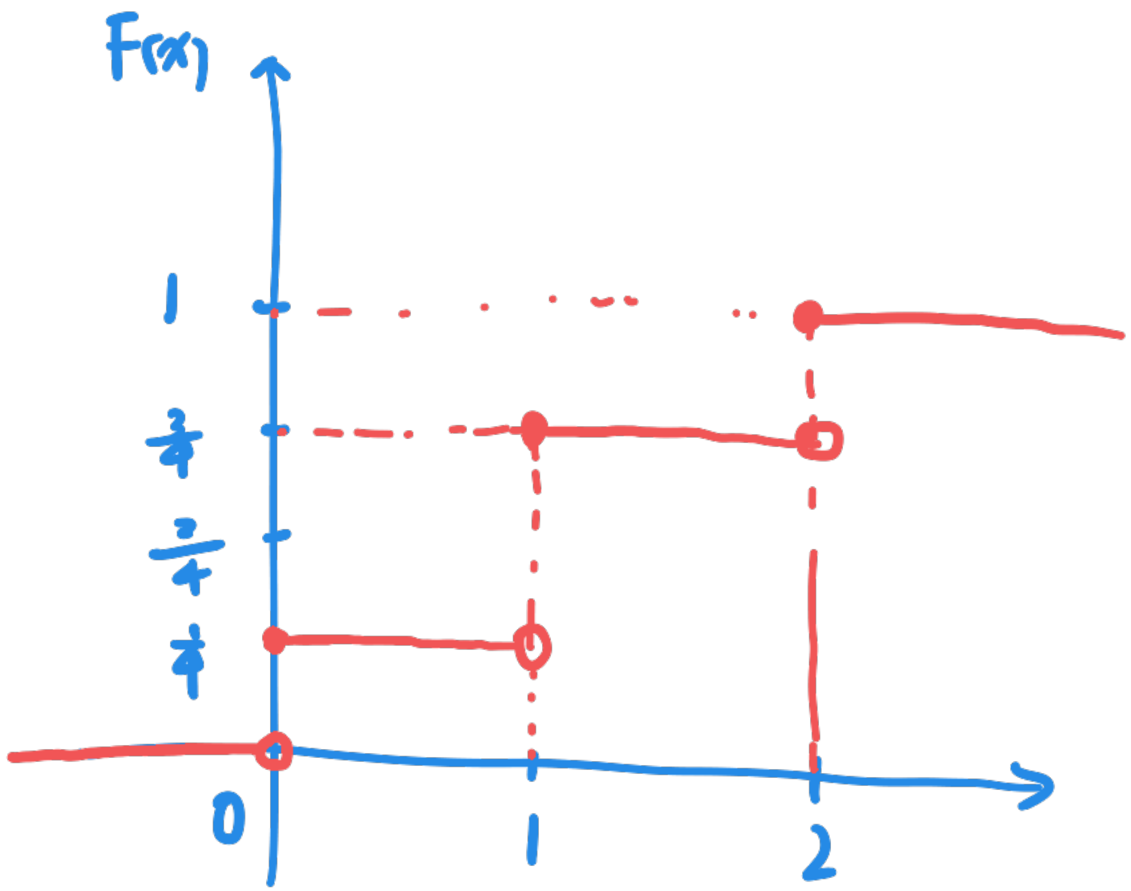
$$= \int_{0}^{x} f_X(x)\,dx$$

$$= \int_{0}^{x} 1\,dx = x \quad , \quad \leftarrow 0 < x < 1$$

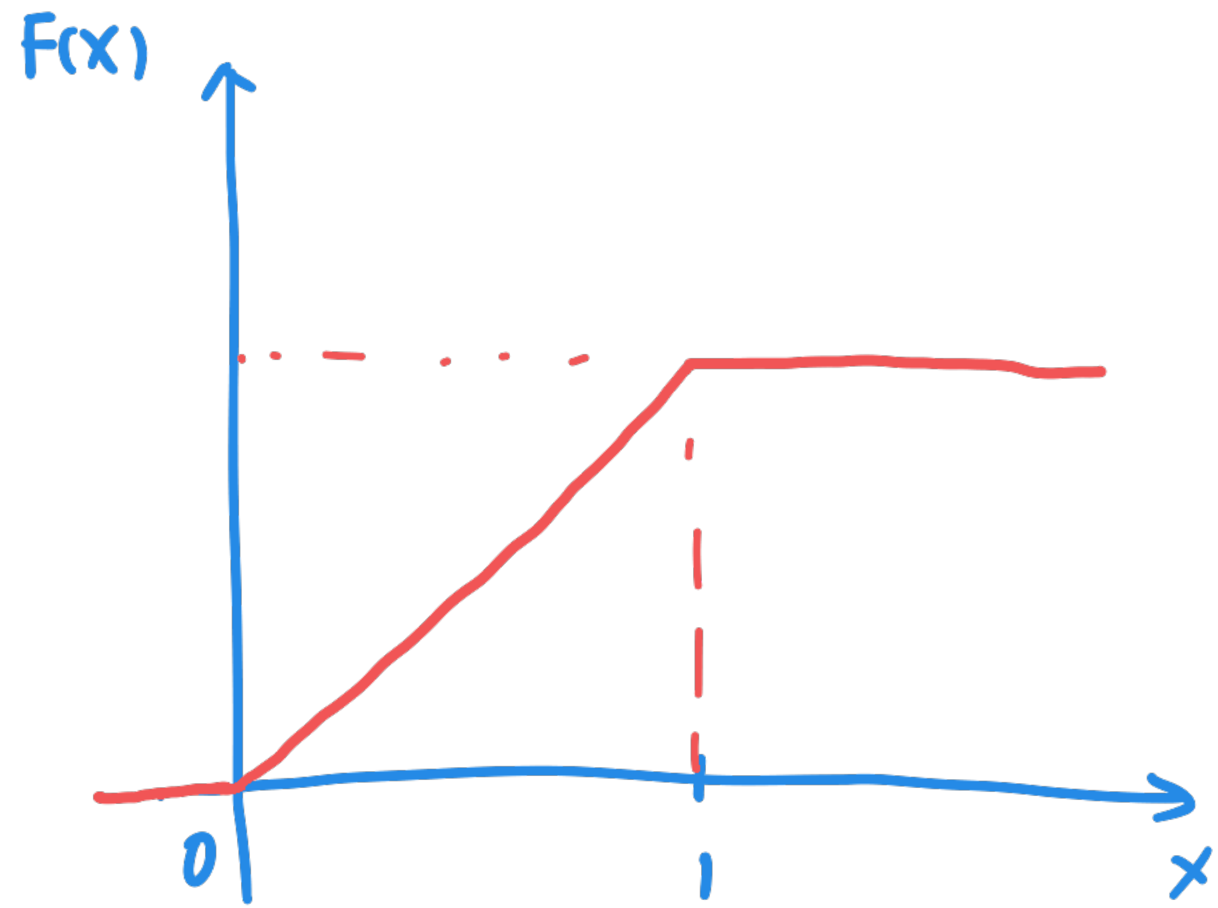


$$\lim_{x \to 0} F_X(x) = 0$$

$$\lim_{x \to 1} F_X(x) = 1$$



Discrete. R.V.



Continuous R.V.

More properties of CDF

- If X is a discrete random variable, $F(x)$ is a right-continuous step function; if X is a continuous random variable, $F(x)$ is a continuous function.
- For a discrete random variable X with support $\mathcal{A} = \{x_1, x_2, \ldots, x_n\}$, where $x_1 < x_2 < \cdots < x_n$,

$$f_X(x_i) = F_X(x_i) - F_x(x_{i-1}), \ \ i = 2, 3, \ldots, n,$$

  and $f_X(x_1) = F(x_1)$.
- For a continuous random variable X, if $f_X(x)$ is continuous, then $F'_X(x)$ exists, and

$$f_X(x) = F'_X s(x).$$

- In actuarial science, biostatistics, and reliability, the related **survivor** function, defined as $S(x) = P(X \cancel{\leq} \geq x)$, is used more frequently than the cumulative distributed function. If X is a patient's survival time after a particular type of surgery, then $S(x)$ can be considered as the probability that the patient **survives** to time x.

Example 3

Find the PMF of the random variable X with CDF given by

$$F_X(x) = \begin{cases} 1/9 & x = 1 \\ 3/9 & x = 2 \\ 5/9 & x = 3 \\ 6/9 & x = 4 \\ 8/9 & x = 6 \\ 1 & x = 9 \end{cases}$$

$A = \{1, 2, 3, 4, 6, 9\}$

$\Rightarrow X$ is discrete

$$f_X(x) = \begin{cases} F_X(1), & x = 1 \\ F_X(2) - F_X(1), & x = 2 \\ F_X(3) - F_X(2), & x = 3 \\ F_X(4) - F_X(3), & x = 4 \\ F_X(6) - F_X(4), & x = 6 \\ F_X(9) - F_X(6) & x = 9 \end{cases} = \begin{cases} 1/9 & x = 1 \\ 3/9 - 1/9 & x = 2 \\ 5/9 - 3/9 & x = 3 \\ 6/9 - 5/9 & x = 4 \\ 8/9 - 6/9 & x = 6 \\ 9/9 - 8/9 & x = 9 \end{cases} = \begin{cases} 1/9 & x = 1 \\ 2/9 & x = 2 \\ 2/9 & x = 3 \\ 1/9 & x = 4 \\ 2/9 & x = 6 \\ 1/9 & x = 9 \end{cases}$$

Example 4

Find the PDF of the random variable X with CDF given by

$$F_X(x) = \frac{x^2}{4}, \quad 0 < x < 2$$

$\mathcal{X} = \{0 < x < 2\}$

$\Rightarrow X$ is continuous.

$$f_X(x) = F_X'(x) = \frac{d\left(\frac{x^2}{4}\right)}{dx} = \frac{2x}{4} = \frac{x}{2} \quad 0 < x < 2$$

Thank You

THANK YOU!