Department of Mathematics
College of William & Mary

univariate R.V.
bivariate R.V.

MATH 451/551

Chapter 6. Joint Distribution

6.1 Bivariate Distribution

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Motivating Examples

Examples

- economics: GDP and unemployment
- sociology: a wife's height and husband's height
- capitalism: college football game — soft drink vs. hot dog sales
- medicine: cholesterol level, triglyceride level, blood pressure

Goal

Extend the probability models for random variables developed so far to two or more random variables.

Bivariate Distribution

Motivating Example

Automobile crashes in U.S. in 2008 involving a fatality.

Y =

X =

Time of day	Number of non-alcohol impaired crashes	Number of alcohol impaired crashes	Total	Percent alcohol impaired crashes
Midnight to 2:59 a.m.	1603	2883	4486	64%
3 a.m. to 5:59 a.m.	1413	1361	2774	49%
6 a.m. to 8:59 a.m.	2768	468	3236	14%
9 a.m. to 11:59 a.m.	2992	293	3285	9%
Noon to 2:59 p.m.	3880	476	4356	11%
3 p.m. to 5:59 p.m.	4269	1056	5325	20%
6 p.m. to 8:59 p.m.	3636	1706	5342	32%
9 p.m. to 11:59 p.m.	2640	2312	4952	47%
Total	23,201	10,555	33,756	

Bivariate Distribution

Motivating Example

Automobile crashes in U.S. in 2008 involving a fatality.

	$Y = 0$	$Y = 1$	Total
$X = 1$	0.047	0.085	0.132
$X = 2$	0.042	0.041	0.083
$X = 3$	0.082	0.014	0.096
$X = 4$	0.088	0.009	0.097
$X = 5$	0.115	0.014	0.129
$X = 6$	0.126	0.031	0.157
$X = 7$	0.108	0.051	0.159
$X = 8$	0.079	0.068	0.147
Total	0.687	0.313	1.000

Bivariate Random Variables

Given a random experiment with an associated sample space S, define the bivariate random variables X and Y that assign to each element $s \in S$ one and only one pair of real numbers $X(s) = x$ and $Y(s) = y$. The support of these random variables is the set of ordered pairs

$$\mathcal{A} = \{(x, y) \mid x = X(s),\ y = Y(s),\ s \in S\}$$

- **Joint Probability Mass Functions:** For some set $A \subset \mathcal{A}$, if $P(A)$ is $P(A) = P\{(X, Y) \in A\} = \sum\sum_A f(x, y)$ when X and Y are discrete random variables, then $f(x, y)$ is the *joint probability mass function (pmf)* of X and Y.
- **Joint Probability Density Functions:** For some set $A \subset \mathcal{A}$, if $P(A)$ is $P(A) = P\{(X, Y) \in A\} = \int\int_A f(x, y)dydx$ when X and Y are continuous random variables, then $f(x, y)$ is the *joint probability density function (pdf)* of X and Y.

Existence Conditions

- **Existence Conditions for Discrete Random Variables X and Y:**

$1 \geq$

$$\sum\sum_{A} f(x,y) = 1 \quad \text{and} \quad f(x,y) \geq 0 \text{ for all real } x \text{ and } y$$

- **Existence Conditions for Continuous Random Variables X and Y:**

$$\int\int_{A} f(x,y)\, dy\, dx = 1 \quad \text{and} \quad f(x,y) \geq 0 \text{ for all real } x \text{ and } y$$

Example 1

52 cards

Deal two cards from a well-shuffled deck. Let the random variable X be the number of aces dealt and let the random variable Y be the number of face cards dealt. Find $f(x, y)$ and calculate the probability that the hand will contain more aces than face cards.

♠A ♡A ♣A ♢A (4 Aces) X & Y Discrete

♠ J Q K ♡ J Q K ♣ J Q K ♢ J Q K (12 face cards)

$$A = \{(x,y) \mid (0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$$

$$f_{X,Y}(x,y) = P(X=x, Y=y)$$

$$f_{X,Y}(0,0) = P(X=0, Y=0) = \frac{\binom{36}{2}}{\binom{52}{2}}$$

$$f_{X,Y}(0,1) = P(X=0, Y=1) = \frac{\binom{12}{1}\binom{36}{1}}{\binom{52}{2}}$$

$$f_{X,Y}(1,1) = P(X=1, Y=1) = \frac{\binom{4}{1}\binom{12}{1}}{\binom{52}{2}}$$

$$f_{X,Y}(x,y) = P(X=x, Y=y)$$

$$= \frac{\binom{4}{x}\binom{12}{y}\binom{36}{2-x-y}}{\binom{52}{2}}, \quad (x,y) \in \mathcal{A}.$$

$P(A)$ where $A = \{(x,y) \mid x = y\}$

$$P(A) = P(X=Y=0) + P(X=Y=1)$$

$$= \frac{\binom{36}{2}}{\binom{52}{2}} + \frac{\binom{4}{1}\binom{12}{1}}{\binom{52}{2}} = 0.5113$$

Example 2

Toss a pair of fair dice. Let X be the smaller number tossed and Y be the larger number tossed. Find

1. the joint probability mass function $f(x, y)$
2. $P(Y = 2X)$

$$S = \left\{ (x,y) \,\middle|\, \begin{array}{l} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,2),(2,3),(2,4),(2,5),(2,6), \\ (3,3),(3,4),(3,5),(3,6), \\ (4,4),(4,5),(4,6),(5,5),(5,6),(6,6) \end{array} \right\} = \left\{ (x,y) \,\middle|\, \begin{array}{l} x=1,2,\cdots,6 \\ y=1,2,\cdots,6, \\ x \le y \end{array} \right\}$$

$$= \{(x,y) \mid x=1,2,\cdots,6,\ y=x,x+1,\cdots,6\}$$

X, Y are Discrete.

$f_{X,Y}(x,y)$

X \ Y	1	2	3	4	5	6
1	1/36	2/36	2/36	2/36	2/36	2/36
2		1/36	2/36	2/36	2/36	2/36
3			1/36	2/36	2/36	2/36
4				1/36	2/36	2/36
5					1/36	2/36
6						1/36

1st (6)	1	1	1	1	1	1	2	2	2	2	2	2	. . .
2nd (6)	1	2	3	4	5	6	1	2	3	4	5	6	

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$$P(Y=2X) = P(X=1, Y=2) + P(X=2, Y=4) + P(X=3, Y=6)$$

$$= \frac{2}{36} \times 3 = \frac{1}{6}$$

Example 3

Jordan and Greta agree to meet at the library between 2:00 PM and 3:00 PM. Their arrival times are independent and uniformly distributed between 2:00 PM and 3:00 PM. If they wait 15 minutes for the other, find the probability that they meet.

X = Arrival time for Jordan $\quad f_X(x) = \frac{1}{60} \quad 0<x<60$

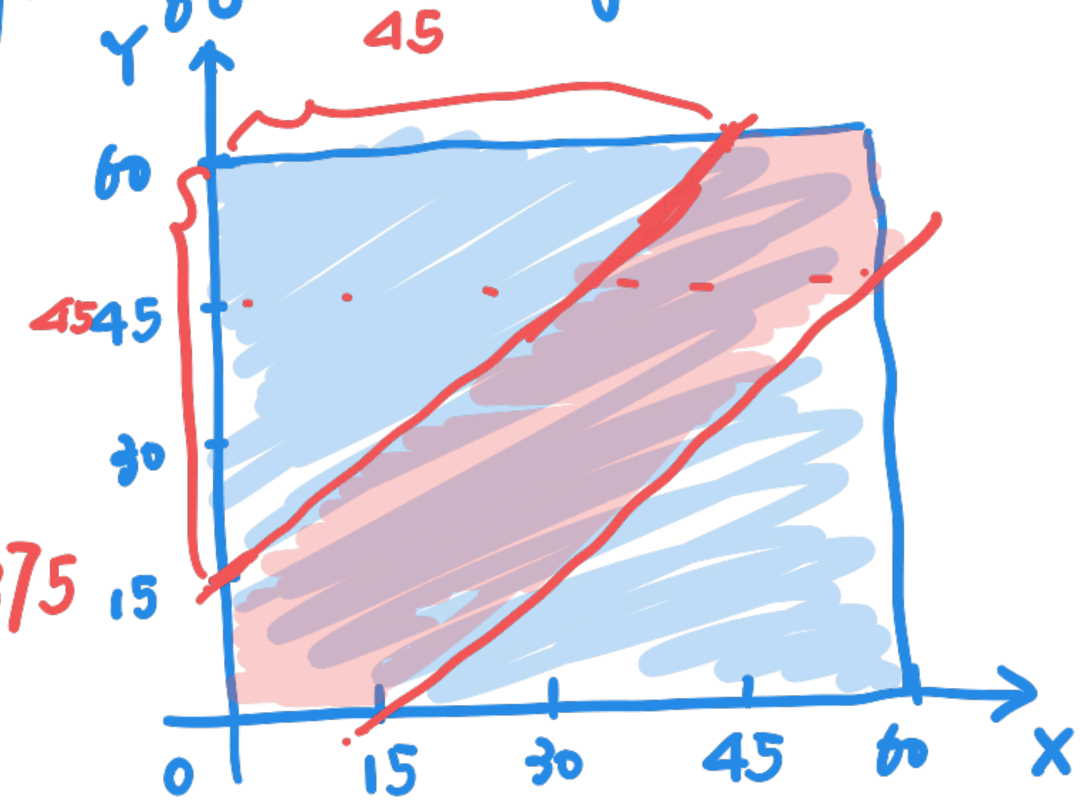
Y = Arrival time for Greta. $\quad f_Y(y) = \frac{1}{60} \quad 0<y<60$

$\mathcal{A} = \{(x,y) \mid 0<x<60,\ 0<y<60\}$

$f_{X,Y}(x,y) = \frac{1}{60\times 60} = \frac{1}{3600}$

$Y - X \le 15 \qquad X - Y \le 15$

$P(|X-Y|\le 15) = \frac{60^2 - \frac{45^2}{2}\cdot 2}{60^2} = \frac{7}{16} = 0.4375$



$$P(|X-Y|\le 15) = 1 - P(|X-Y|>15)$$

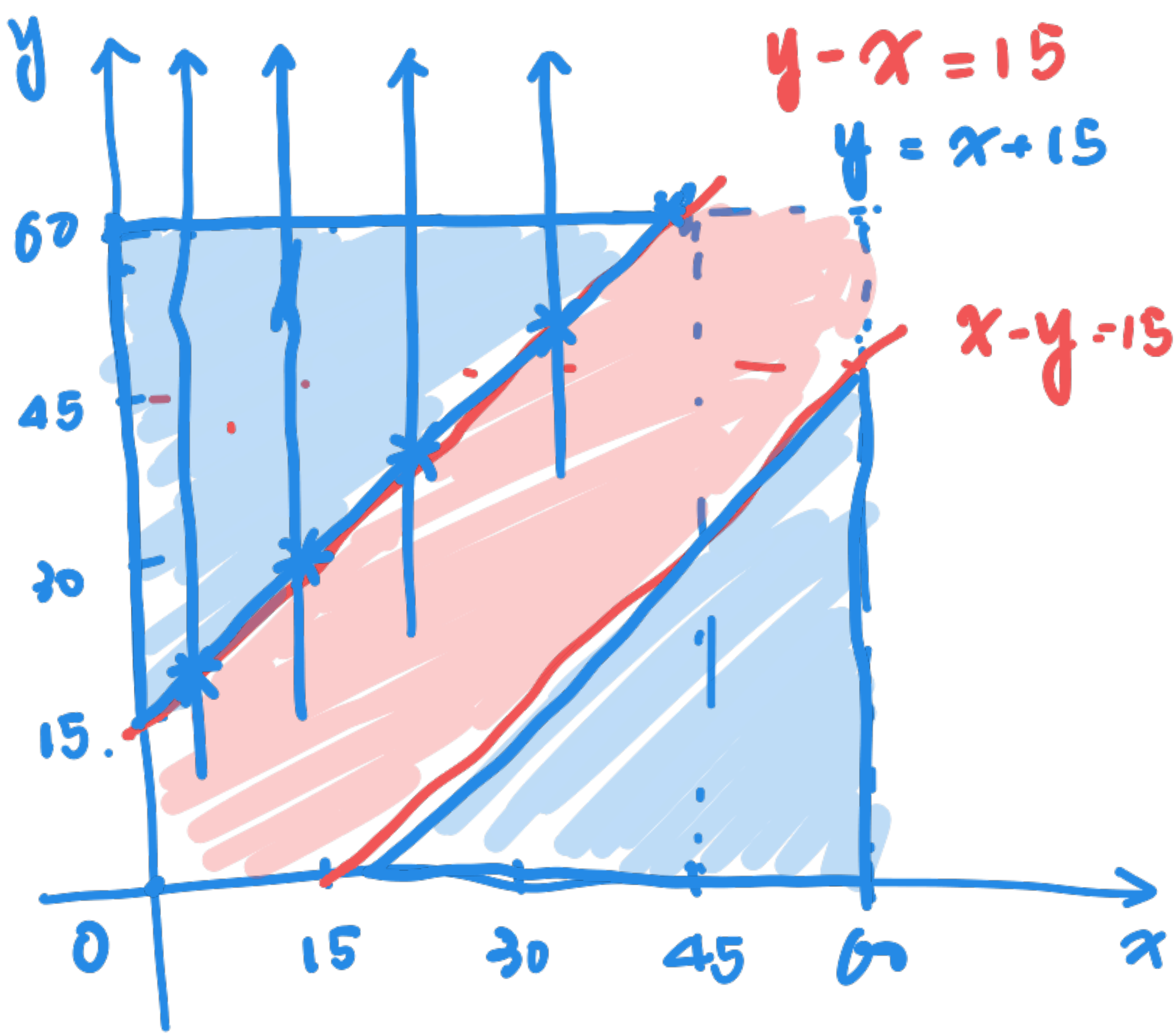
$$= 1 - 2\times \underbrace{P(Y-X>15)}$$

$$= 1 - 2\times \int_0^{45}\int_{x+15}^{60} \frac{1}{3600}\,dy\,dx$$

$$= 1 - 2\times \int_0^{45} \frac{\overset{45}{\cancel{60}-x-\cancel{15}}}{3600}\,dx$$

$$= 1 - 2\times\left(\frac{45(45-0)}{3600} - \frac{45^2-0^2}{2\times 3600}\right)$$

$$= 1 - \cancel{2}\times\frac{45^2}{\cancel{2}\times 3600} = \frac{7}{16}$$



Example 4

Let the continuous random variables X and Y have joint probability density function

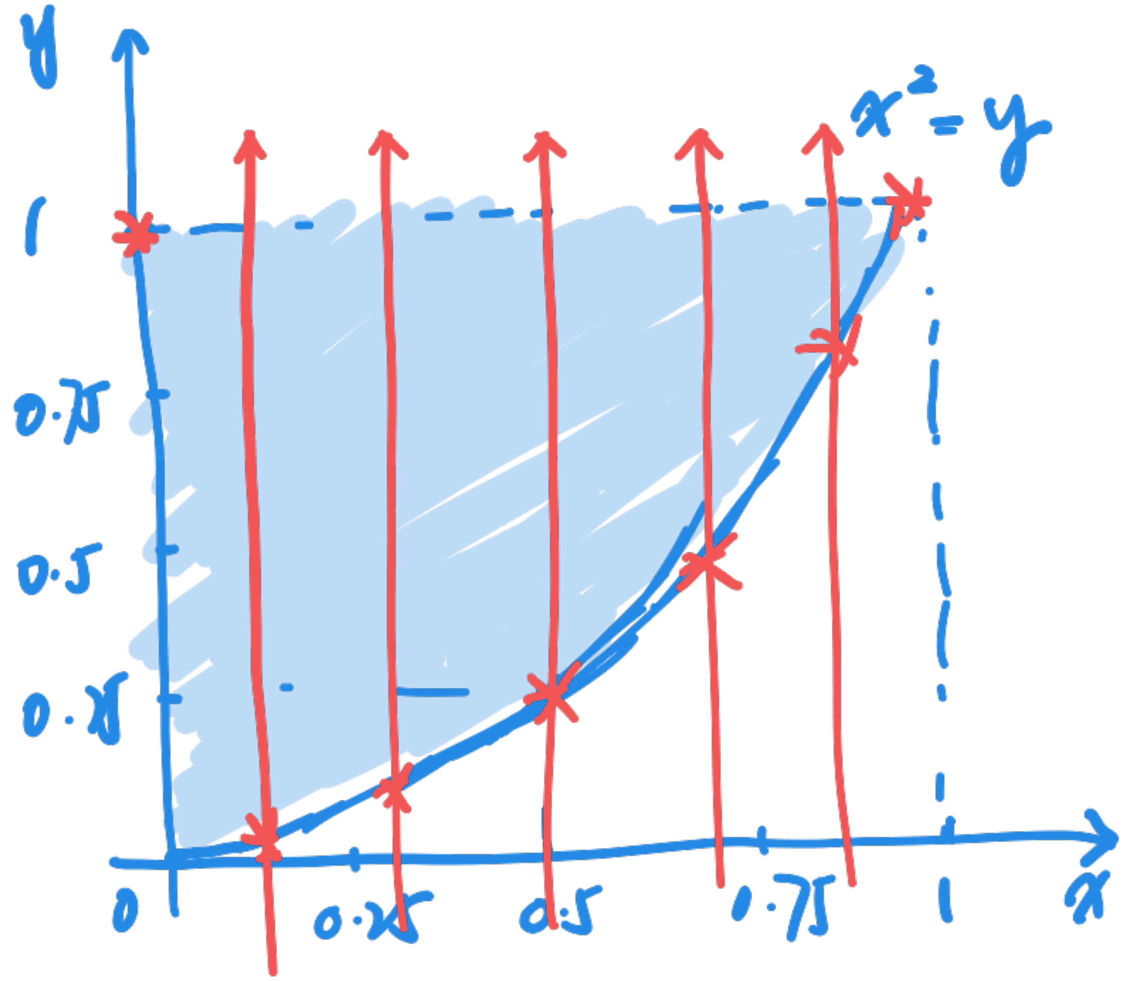
$$f(x, y) = 1, \quad 0 < x < 1,\ 0 < y < 1$$

Find $P\left(0 < X^2 < y < 1\right)$.

$$P(0<x^2<y<1) = \int_0^1 \int_{x^2}^1 f(x,y)\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^1 1\, dy\, dx = \int_0^1 (1-x^2)\, dx$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$



R Functions

Function	Returned Value
dunif(x, a, b)	calculates the probability density function $f(x)$
punif(x, a, b)	calculates the cumulative distribution function $F(x)$
qunif(u, a, b)	calculates the percentile (quantile) $F^{-1}(u)$
runif(m, a, b)	generates m random variates

Example 1

A shuttle train at a busy airport completes a circuit between two terminals every five minutes. What is the probability that a passenger will wait more than three minutes for a shuttle train?

Example 2

What is the probability that the quadratic equation $x^2 + Bx + 1 = 0$ has two real roots, where $B \sim U(0, 3)$? (Hint: The quadratic equation $ax^2 + bx + c = 0$ has two real roots if the discriminant $b^2 - 4ac$ is positive.)

Example 3

Let $X \sim U(0,1)$, find $V(3\lfloor 2X \rfloor + 4)$.

Example 4

Divide a line segment of unit length randomly into two parts. Find the expected value of the product of the lengths of the two segments.

THANK YOU!